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HEAT CONDUCTION AND HEAT TRANSFER IN TECHNOLOGICAL PROCESSES

MATHEMATICAL MODEL OF HEATING A PRISM WITH BOUNDARY CONDITIONS OF THE 3RD KIND

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This paper describes the procedure of computational determination of the temperature field of a prismatic workpiece heated in a continuous furnace with account for the temperature dependence of the thermal diffusivity. For a numerical solution of the two-dimensional heat conduction equation with boundary conditions of the 3rd kind, an implicit scheme has been used. The calculated time dependences of the temperature for three characteristic points of the cross-section of the prismatic steel workpiece have been compared to the experimental data. The heat transfer coefficients at which the experimental data and the calculated values have a minimum discrepancy have been determined.

Introduction. It is known that the thermophysical and mechanical characteristics of metals vary with changing temperature. Thus, according to the thermometric data for steel 20 workpieces before rolling [1], the thermal diffusivity decreases by a factor of 3 upon heating from 500 to 1000 K. The existence of pronounced temperature dependences of the basic thermophysical and deformation-strength characteristics of structural materials (specific heat capacity, heat conductivity coefficient, thermal diffusivity, ultimate and yield strengths, Young's modulus, etc.) leads to the necessity of refining the estimates of temperatures and thermal stresses with the aim of saving energy and upgrading the quality of metal products.

Taking into account the rather complex form of the temperature dependences of the above indices, their approximations in the form of piecewise linear functions are used [2]. Numerical solution of such problems entails taking into account the nonlinearity of the differential equation and the boundary conditions, due to which it is necessary to use grid methods in which the temperature distributions inside the region and on its surface are given at nodal points. The change in the thermophysical coefficients in the process of heating is determined by referring them to the temperatures at the nodal points at the beginning of the time intervals and checking the stability condition. As applied to the conditions of continuous casting of workpieces, the numerical solution can be simplified by passing to a two-dimensional formulation of the problem, since the length of the ingot formed exceeds many times the dimensions of its cross-section.

As an example, we consider below the conditions of heating in a continuous furnace a prismatic steel workpiece with heat exchange by the Newton law. Identification (adjustment) of the heating model is carried out by correcting the heat transfer coefficients proceeding from comparison of the experimental and calculated time dependences of temperature at control points.

Formulation of the Problem. Let us consider the mathematical formulation of the problem on the heat propagation in the cross-section of a steel workpiece heated in a continuous furnace.

The quasi-linear heat conduction equation with variable thermal diffusivity [1] is of the form

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$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(a^2(u)\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(a^2(u)\frac{\partial u}{\partial y}\right);$$

the boundary conditions are $u(x, y, 0) = u_0 = 393$ K; the boundary conditions of the 3rd kind modeling the heat propagation by the Newton law are

$$\left.\frac{\partial u(x, y, t)}{\partial n}\right|_{\Gamma}=\alpha_1\left(u(x, y, t)-\theta(t)\right).$$

Here $\partial/\partial n$ is a derivative with respect to the inner normal to the workpiece edge, 1/m.

We make a substitution $v(x, y, t) = \frac{1}{a^2(u_0)u_0}\int\limits_{u_0}^{u(x,y,t)} a^2(u)du$. In particular, the dependence of the thermal diffusivity with a dimensionally of one square meter per second for the material steel St3, according to [3], can be approximated by three linear parts:

$$a^2(u)=\begin{cases}1.86\cdot10^{-5}-1.45\cdot10^{-8}u, & 373\text{ K}\le u\le 1023\text{ K};\\ -1.416\cdot10^{-5}+0.775\cdot10^{-8}u, & 1023\text{ K}\le u\le 1273\text{ K};\\ 5.7\cdot10^{-6}, & 1273\text{ K}\le u\le 1423\text{ K}.\end{cases}$$

Integrating and solving the square equation, we obtain the dependence of the dimensionless temperature on the new variable v:

$$u(v)=\begin{cases}1+\dfrac{a^2(u_0)}{a_0u_0}\left(1-\sqrt{1-\dfrac{2va_0u_0}{a^2(u_0)}}\right), & v\in[0;1.036];\\ -\dfrac{A}{Bu_0}+\sqrt{\dfrac{A^2}{B^2u_0^2}+\dfrac{2a^2(u_0)}{Bu_0}(v-1.07)}, & v\in[1.036;1.27];\\ 2.27v+0.37, & v\in[1.27;1.439].\end{cases}$$

Differentiating the dependence obtained with respect to the variable v, we get the expression

$$u'(v)=\begin{cases}\dfrac{1}{\sqrt{1-\dfrac{2va_0u_0}{a^2(u_0)}}}, & v\in[0;1.036];\\ \dfrac{a^2(u_0)}{\sqrt{A^2+2a^2(u_0)u_0B(v-1.07)}}, & v\in[1.036;1.27];\\ 2.27, & v\in[1.27;1.439].\end{cases}$$

The substitution made permits formulating the problem as

$$\varphi'(v)\frac{\partial v}{\partial\tau}=\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2},\quad v(x, y, 0)=0,$$

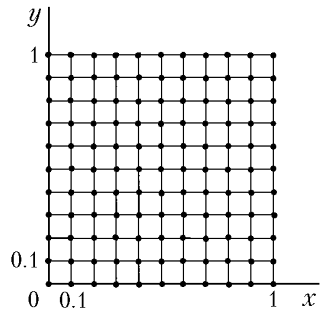

Fig. 1. Discretization scheme of the workpiece cross-section.

with the boundary conditions of the 3d kind $\left.\dfrac{\partial u(x, y, \tau)}{\partial n}\right|_{\Gamma} = \alpha_1(u(x, y, \tau) - \theta(\tau))$.

To reduce the number of variables, it is expedient to bring the above mathematical formulation of the problem into a dimensionless form. To this end, we used the following characteristic constants: the initial temperature of the workpiece u_0 = 393 K, the length of the cross-section side of the workpiece l_0 = 0.125 m, the characteristic time t_0 = $9\cdot10^4$ sec, and $a^2(u_0) = 1.29\cdot10^{-5}$ m^2/sec. As the final result, the problem is written in the form

$$\varphi'(v)\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}, \quad v(x, y, 0) = 0, \quad x, y \in [0; 1], \quad t \in [0; 0.04];$$

$$\left.\frac{\partial u(v(x, y, t))}{\partial n}\right|_{\Gamma} = \alpha_1(u(v(x, y, t)) - \theta(t)), \quad \theta(t) = 2.6 + 22.33t;$$

$$\varphi'(v) = \frac{l_0^2}{t_0}\frac{1}{a^2(u_0 u(v))}.$$

Construction of the Grid Analog. To obtain a numerical solution, consider the grid analog of the heat conduction equation at $i, j = 0, \ldots, 10$ obtained by the discretization of the cross-section of a prismatic workpiece in the form of 10 × 10 partitions into parts of constant length with the number of nodes 11 × 11 (Fig. 1). We shall further use the discrete analog of the dimensionless temperature and finite-difference schemes with weights [4, 5]:

$$\varphi'(y_{i,j}^{t+1})\frac{y_{i,j}^{t+1} - y_{i,j}^{t}}{\Delta\tau} = \Lambda_x\left(\sigma y_{i,j}^{t+1} + (1-\sigma)y_{i,j}^{t}\right) + \Lambda_y\left(\sigma y_{i,j}^{t+1} + (1-\sigma)y_{i,j}^{t}\right).$$

Here Λ_x is the difference analog of the second derivative, which is calculated by the formula

$$\Lambda_x y_{i,j}^{t} = \frac{y_{i+1,j}^{t} - 2y_{i,j}^{t} + y_{i-1,j}^{t}}{h^2}.$$

For the best approximation [6] of the sought function by the discrete analog, we assume $\sigma = 0.5$. The finite-difference scheme of the solution has the form

$$\varphi'(y_{i,j}^{t+1})\frac{y_{i,j}^{t+1} - y_{i,j}^{t}}{\Delta\tau} = \frac{1}{2}\left(\frac{y_{i+1,j}^{t+1} - 2y_{i,j}^{t+1} + y_{i-1,j}^{t+1}}{h^2}\right) + \frac{1}{2}\left(\frac{y_{i+1,j}^{t} - 2y_{i,j}^{t} + y_{i-1,j}^{t}}{h^2}\right)$$

$$+\frac{1}{2}\left(\frac{y_{i,j+1}^{t+1}-2y_{i,j}^{t+1}+y_{i,j-1}^{t+1}}{h^2}\right)+\frac{1}{2}\left(\frac{y_{i,j+1}^{t}-2y_{i,j}^{t}+y_{i,j-1}^{t}}{h^2}\right).$$

For the approximation of the boundary conditions, we use the following difference analogs:

$$\sigma\left(u'(y_{0,j}^{t+1})\frac{y_{1,j}^{t+1}-y_{0,j}^{t+1}}{h}-\alpha_1 u(y_{0,j}^{t+1})\right)+(1-\sigma)\left(u'(y_{0,j}^{t})\frac{y_{1,j}^{t}-y_{0,j}^{t}}{h}-\alpha_1 u(y_{0,j}^{t})\right)$$

$$=\frac{1}{2}h\frac{l_0^2}{t_0}u'(y_{0,j}^{t})\frac{y_{0,j}^{t+1}-y_{0,j}^{t}}{\Delta\tau}-\alpha_1\theta\left(\left(t+\frac{1}{2}\right)\tau\right);$$

$$\sigma\left(u'(y_{i,0}^{t+1})\frac{y_{i,1}^{t+1}-y_{i,0}^{t+1}}{h}-\alpha_1 u(y_{i,0}^{t+1})\right)+(1-\sigma)\left(u'(y_{i,0}^{t})\frac{y_{i,1}^{t}-y_{i,0}^{t}}{h}-\alpha_1 u(y_{i,0}^{t})\right)$$

$$=\frac{1}{2}h\frac{l_0^2}{t_0}u'(y_{i,0}^{t})\frac{y_{i,0}^{t+1}-y_{i,0}^{t}}{\Delta\tau}-\alpha_1\theta\left(\left(t+\frac{1}{2}\right)\tau\right);$$

$$\sigma\left(-u'(y_{10,j}^{t+1})\frac{y_{10,j}^{t+1}-y_{9,j}^{t+1}}{h}-\alpha_1 u(y_{10,j}^{t+1})\right)+(1-\sigma)\left(-u'(y_{10,j}^{t})\frac{y_{10,j}^{t}-y_{9,j}^{t}}{h}-\alpha_1 u(y_{10,j}^{t})\right)$$

$$=\frac{1}{2}h\frac{l_0^2}{t_0}u'(y_{10,j}^{t})\frac{y_{10,j}^{t+1}-y_{10,j}^{t}}{\Delta\tau}-\alpha_1\theta\left(\left(t+\frac{1}{2}\right)\tau\right);$$

$$\sigma\left(-u'(y_{i,10}^{t+1})\frac{y_{i,10}^{t+1}-y_{i,9}^{t+1}}{h}-\alpha_1 u(y_{i,10}^{t+1})\right)+(1-\sigma)\left(-u'(y_{i,10}^{t})\frac{y_{i,10}^{t}-y_{i,9}^{t}}{h}-\alpha_1 u(y_{i,10}^{t})\right)$$

$$=\frac{1}{2}h\frac{l_0^2}{t_0}u'(y_{i,10}^{t})\frac{y_{i,10}^{t+1}-y_{i,10}^{t}}{\Delta\tau}-\alpha_1\theta\left(\left(t+\frac{1}{2}\right)\tau\right).$$

As in the case of the grid analog of the heat conduction equation, for the best approximation of the boundary conditions we assume $\sigma = 0.5$. By simple mathematical manipulations, we get from the grid analog of the heat conduction equation the following system of linear algebraic equations:

$$\frac{\tau}{2h^2}\left(y_{i+1,j}^{t+1}+y_{i-1,j}^{t+1}+y_{i,j+1}^{t+1}+y_{i,j-1}^{t+1}\right)-\left(\frac{2\tau}{h^2}+\varphi'(y_{i,j}^{t+1})\right)y_{i,j}^{t+1}=\left(\frac{2\tau}{h^2}-\varphi'(y_{i,j}^{t+1})\right)y_{i,j}^{t}$$

$$-\frac{\tau}{2h^2}\left(y_{i+1,j}^{t}+y_{i-1,j}^{t}+y_{i,j+1}^{t}+y_{i,j-1}^{t}\right),\quad i,j=1,\ldots,9\,.$$

Likewise, we obtain a system of linear equations from the grid analog of the boundary conditions of the 3d kind

$$u'(y_{0,j}^{t+1})\,y_{1,j}^{t+1}-\left(\frac{h^2}{\tau}\frac{l_0^2}{t_0}u'(y_{0,j}^{t})+u'(y_{0,j}^{t+1})\right)y_{0,j}^{t+1}=\alpha_1 h\left(u(y_{0,j}^{t+1})+u(y_{0,j}^{t})\right)+u'(y_{0,j}^{t})\,y_{1,j}^{t}$$

$$+\left(u'(y_{0,j}^{t})-\frac{h^2}{\tau}\frac{l_0^2}{t_0}u'(y_{0,j}^{t})\right)y_{0,j}^{t}-2\alpha_1 h\theta\left(\left(t+\frac{1}{2}\right)\tau\right),\quad j=0,\ldots,10\,;$$

$$u'\left(y_{i,0}^{t+1}\right) y_{i,1}^{t+1} - \left(\frac{h^2}{\tau}\frac{l_0^2}{t_0} u'\left(y_{i,0}^{t}\right) + u'\left(y_{i,0}^{t+1}\right)\right) y_{i,0}^{t+1} = \alpha_1 h \left(u\left(y_{i,0}^{t+1}\right) + u\left(y_{i,0}^{t}\right)\right) + u'\left(y_{i,0}^{t}\right) y_{i,1}^{t}$$

$$+ \left(u'\left(y_{i,0}^{t}\right) - \frac{h^2}{\tau}\frac{l_0^2}{t_0} u'\left(y_{i,0}^{t}\right)\right) y_{i,0}^{t} - 2\alpha_1 h \theta\left(\left(t+\frac{1}{2}\right)\tau\right), \quad j = 1, \ldots, 9\ ;$$

$$\left(\frac{h^2}{\tau}\frac{l_0^2}{t_0} u'\left(y_{10,j}^{t}\right) + u'\left(y_{10,j}^{t+1}\right)\right) y_{10,j}^{t+1} - u'\left(y_{10,j}^{t+1}\right) y_{9,j}^{t+1} = u'\left(y_{10,j}^{t}\right) y_{9,j}^{t} - \left(u'\left(y_{10,j}^{t}\right) - \frac{h^2}{\tau}\frac{l_0^2}{t_0} u'\left(y_{10,j}^{t}\right)\right) y_{10,j}^{t}$$

$$- \alpha_1 h \left(u\left(y_{10,j}^{t+1}\right) + u\left(y_{10,j}^{t}\right)\right) + 2\alpha_1 h \theta\left(\left(t+\frac{1}{2}\right)\tau\right), \quad j = 0, \ldots, 10\ ;$$

$$\left(\frac{h^2}{\tau}\frac{l_0^2}{t_0} u'\left(y_{i,10}^{t}\right) + u'\left(y_{i,10}^{t+1}\right)\right) y_{i,10}^{t+1} - u'\left(y_{i,10}^{t+1}\right) y_{i,9}^{t+1} = u'\left(y_{i,10}^{t}\right) y_{i,9}^{t} - \left(u'\left(y_{i,10}^{t}\right) - \frac{h^2}{\tau}\frac{l_0^2}{t_0} u'\left(y_{i,10}^{t}\right)\right) y_{i,10}^{t}$$

$$- \alpha_1 h \left(u\left(y_{i,10}^{t+1}\right) + u\left(y_{i,10}^{t}\right)\right) + 2\alpha_1 h \theta\left(\left(t+\frac{1}{2}\right)\tau\right), \quad j = 1, \ldots, 9\ .$$

To solve the obtained nonlinear system of equation, we use the iteration method of [4, 5] substituting into the nonlinear part the known values of $\overset{(s)}{y}{}_{i,j}^{t+1}$ and into the linear parts, accordingly, $\overset{(s+1)}{y}{}_{i,j}^{t+1}$. As a result, we obtain a system including 121 algebraic equations for $\overset{(s+1)}{y}{}_{i,j}^{t+1}$ (of these, 81 equations correspond to the approximation of the quasi-linear heat conduction equation and 40 equations form a grid analog of the boundary conditions of the 3d kind). We solve the system and find a numerical solution for the $(s + 1)$ iteration.

The system of linear algebraic equations obtained form the grid analog of the quasi-linear heat conduction equation has the form

$$\frac{\tau}{2h^2}\left(\overset{(s+1)}{y}{}_{i+1,j}^{t+1} + \overset{(s+1)}{y}{}_{i-1,j}^{t+1} + \overset{(s+1)}{y}{}_{i,j+1}^{t+1} + \overset{(s+1)}{y}{}_{i,j-1}^{t+1}\right) - \left(\frac{2\tau}{h^2} + \varphi'\left(\overset{(s)}{y}{}_{i,j}^{t+1}\right)\right) \overset{(s+1)}{y}{}_{i,j}^{t+1} = \left(\frac{2\tau}{h^2} - \varphi'\left(\overset{(s)}{y}{}_{i,j}^{t+1}\right)\right) y_{i,j}^{t}$$

$$- \frac{\tau}{2h^2}\left(y_{i+1,j}^{t} + y_{i-1,j}^{t} + y_{i,j+1}^{t} + y_{i,j-1}^{t}\right), \quad i, j = 1, \ldots, 9\ .$$

The equations found from one of the boundary conditions for the $(s + 1)$ iteration are written as follows:

$$u'\left(\overset{(s)}{y}{}_{0,j}^{t+1}\right) \overset{(s+1)}{y}{}_{1,j}^{t+1} - \left(\frac{h^2}{\tau}\frac{l_0^2}{t_0} u'\left(y_{0,j}^{t}\right) + u'\left(\overset{(s)}{y}{}_{0,j}^{t+1}\right)\right) \overset{(s+1)}{y}{}_{0,j}^{t+1} = \alpha_1 h \left(u\left(\overset{(s)}{y}{}_{0,j}^{t+1}\right) + u\left(y_{0,j}^{t}\right)\right)$$

$$+ u'\left(y_{0,j}^{t}\right) y_{1,j}^{t} + \left(u'\left(y_{0,j}^{t}\right) - \frac{h^2}{\tau}\frac{l_0^2}{t_0} u'\left(y_{0,j}^{t}\right)\right) y_{0,j}^{t} - 2\alpha_1 h \theta\left(\left(t+\frac{1}{2}\right)\tau\right), \quad j = 0, \ldots, 10\ .$$

We assume the values of the s-iteration to be known and the values of the zeroth iteration to be equal to the values in the previous layer. The initial conditions are: $y_{i,j}^{0} = 0$, where i, j = 0, ..., 10. Identification of the heating model was carried out by correcting the heat transfer coefficients through minimization of the deviation of the experimental and calculated time dependences of temperature at control points of the workpiece cross-section.

Numerical Approbation. The temperature field of the prism with a cross-section of 0.125×0.125 m^2 was calculated by means of the Mathcad 2001 program. To reduce the grid equations to a system of algebraic equations,

TABLE 1. Reduced External Heat Transfer Coefficients Depending on the Heating Time

Time interval, min	α_1	α_2	α_3	α_4
0	0.025	0.17	0.016	0.15
6	0.08	0.33	0.04	0.27
12	0.17	0.39	0.12	0.3
18	0.24	0.68	0.21	0.54
24	0.24	0.79	0.17	0.61
30	0.27	0.95	0.17	0.62
36	0.48	1.29	0.32	0.55
42	0.46	1.15	0.49	0.41
48	0.51	1.35	0.58	0.23
54	0.65	1.41	0.63	0.34

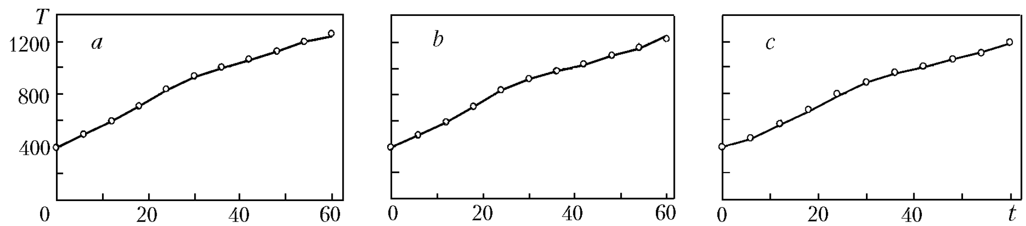

Fig. 2. Time dependence of the temperature in the middle of the upper bound (a), for the angular point of the lower bound (b), and for the center of the lower bound (c). Curves and dots show, respectively, the calculation and the experiment.

the grid values were transformed to a vector by the formula $x_{11i+j} = y_{i,j}^{t}$. The system of algebraic equations with a determinant other than zero was solved by the matrix method described, e.g., in [6]. We used the reduced external heat transfer coefficients for four faces of the workpiece α_1, α_2, α_3, α_4, where α_1 corresponds to the left lateral face, α_2 — to the upper bound, α_3 — to the right lateral face, and α_4 — to the lower bound. We specified the space-time discretization parameters $h = 0.1$, $\Delta\tau = 0.004$, the initial temperature of the workpiece $u_0 = 393$ K, and the manner of the temperature change in the furnace in the dimensionless form $\theta(t) = 2.6 + 22.33\tau$. In the considered example of calculation, to obtain a deviation of the values at steps $(s+1)$ and (s) not exceeding 1%, no more than three iterations were needed. The values of the above coefficients for instants of time in the time interval of heating 0–60 min with a step of 6 min are given in Table 1.

A comparison of the results of the calculations to the experimental values of the temperature at control points of the cross-section is shown in Fig. 2. It should be noted that there is a discrepancy between the values of the external heat transfer coefficients for the left and right lateral faces of the cross-section of a geometrically symmetric workpiece, which can be explained by the nonidentity of the heat flows from on the left and on the right under real heating conditions.

In using the above method of adjustment, the deviation of the calculated temperature values from the experimental data in the workpiece heating time interval of 60 min does not exceed 2% at edge points and 10% in the center of the cross-section.

To calculate the temperature field in the following heating zones, it is necessary to use the corresponding dependences for the furnace temperature and the new initial conditions in the form of the temperature values found in the previous step by the algorithm described above. Taking into account the existing experimental data, adjustment of the heating model is carried out by recalculating the reduced external heat transfer coefficients on each face of the workpiece cross-section.

Conclusions. The proposed method for numerical solution of the quasi-linear heat conduction equation with boundary conditions of the 3d kind makes it possible to find with a reasonable accuracy the temperature field in heating a prismatic steel workpiece with account for the temperature dependence of the thermal diffusivity. The proposed procedure of identification of the developed mathematical model on the basis of minimization of the deviation of calculation results from the experimental data permits overcoming difficulties of calculating heating conditions connected with the specification of external heat transfer coefficients.

NOTATION

$a_0 = 1.45 \cdot 10^{-8}$, modulus of the proportionality coefficient in the temperature dependence of thermal diffusivity in the 373–1023 K range, $m^2/(sec \cdot K)$; $a^2(u)$, temperature dependence of the thermal diffusivity, m^2/sec; $a^2(u_0)$, thermal diffusivity of the material at temperature u_0, m^2/sec; $A = -0.416 \cdot 10^{-5}$, constant in the temperature dependence of the thermal diffusivity in the 1023–1273 K range, m^2/sec; $B = 0.755 \cdot 10^{-8}$, proportionality coefficient in the temperature dependence of the thermal diffusivity in the 1023–1273 K, $m^2/(sec \cdot K)$; h, steps of space coordinate discretization; l_0, characteristic length, m; n, inner normal to the bound of the workpiece cross-section, m; t, time reduced coordinate, min; t_0, characteristic time, sec; u, temperature, K; u_0, initial temperature, K; $u(x, y, t)$, coordinate and time dependence of temperature, K; v, variable; x, y, coordinates in the sectional plane, m; y_{ij}^t, discrete analog of the function of dimensionless temperature in the node ij at a reduced instant of time $t \times \tau$; α, reduced external heat transfer coefficient; Γ, bound of the workpiece cross section; $\theta(t) = 1023 \text{ K} + 0.0975t$, low of furnace temperature change, K; Λ_y, difference analog of the second derivative with respect to the y coordinate; σ, weight factor; τ, time reduced coordinate; $\Delta\tau$, time step of discretization. Subscripts: i, line number of the discrete analog; j, column number of the discrete analog; $'$, sign of the derivative.

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